

USING A SMOOTHING AVERAGING OPERATOR TO EVALUATE MACROSCOPIC PARAMETERS IN STRUCTURALLY INHOMOGENEOUS MATERIALS

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A smoothing averaging operator is used in passing from structural to macroscopic modeling of the stress-strained state of an article from a composite material taking into account finite strains. A model is constructed using an integral operator, in which the macroscopic laws of conservation of energy and mass and the equation of motion have the ordinary form used to describe processes in homogeneous materials. As an example, macroscopic parameters are evaluated in a system consisting of an ensemble of inclusions in an infinite matrix.

Key words: *integral operator, averaging, macroscopic parameters, structurally inhomogeneous materials.*

Introduction. Conversion from modeling of the properties of a medium at the structural level to modeling of the properties at the macroscopic level can be implemented using the hypothesis of macroscopic definability (in which each point of an article is associated with a composite sample of large size in a macroscopically homogeneous state). This approach to constructing macroscopic models involves the following difficulties. Averaging of the state variables of a material (without a detailed analysis of the results of averaging of the constitutive equations for the material in a macroscopically inhomogeneous state) does not allow one to establish whether higher level concepts are correctly introduced and whether the conservation laws are obeyed in determining macroscopic quantities. It is difficult to establish which deformation measure and which stress tensor (Cauchy, Piola, Kirchhoff, etc.) should be averaged.

Use of the operator method to pass from the constitutive equations of the structural level to macroscopic constitutive equations allows one to analyze the properties of both the operator itself and the macroscopic model obtained as a result of applying this operator. In this approach, the case in point is an analysis of the macroscopic model obtained rather than a correct formulation of a hypothesis (choice of a method for calculating the macro-properties and the form of macroscopic equations). Integral averaging operators of the constitutive equations of a medium have been widely used to describe macroscopic processes in multicomponent systems [1–4].

At present, various theories of integral operators have been proposed and the possible regions of their application have been explored. It has been shown that wavelet analysis [5, 6] is an effective means of image filtering intended to distinguish structures of a specified scale level. Using the resolvent-point imaging technique [7–10], it is possible to construct effective meshless algorithms of solving boundary-value problems.

In the present study, we propose to use an integral averaging operator with a smoothly varying differentiable kernel. It can be applied to obtain smooth macroscopic images of fields of structural state variables of a material. An algorithm for examining the average properties of a medium is given below. Initially, it is elucidated why a particular smoothed quantity can be called macroscopic and whether it can be correctly used in the macroscopic laws of conservation of energy and mass and equations of motion. Next, it is checked whether function relations can be established between smoothed parameters. If such relations exist, smoothed state variables or their combinations

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should be called macroscopic state variables and the function relations should be called the macroscopic properties of the medium.

Operator of Evaluation of the Macroscopic State Variables of a Material. An operator that sets up a correspondence between a state function at the structural level $a(t, \mathbf{x})$ and a value $\hat{a}(t, \hat{\mathbf{x}})$ by means of a map

$$\hat{a}(t, \hat{\mathbf{x}}) = \int_{V-S} a(t, \mathbf{x}) \Phi(\hat{\mathbf{x}} - \mathbf{x}) dV$$

will be called the operator of evaluation of the macroscopic characteristic $\hat{a}(t, \hat{\mathbf{x}})$ of the composite material at the point of space $\hat{\mathbf{x}}$ at time t . Here V is a volume of Euclidean space whose points are determined by the radius vector \mathbf{x} , $\hat{\mathbf{x}}$ is the radius vector of a point of space for which the value of the macroscopic quantity is determined. It is assumed that the kernel of the integral vanishes with distance from the point $\hat{\mathbf{x}}$ exceeding the value of a_Φ and is a continuous differentiable function that satisfies the normalization condition $\int_{V-S} \Phi(\hat{\mathbf{x}} - \mathbf{x}) dV = 1$ and the

requirement that it does not depend on the orientation of basis vectors in space. In integration over the volume V , the domain of integration does not include the points of the interface S of the structurally inhomogeneous medium at which the value of the integrand $a(t, \mathbf{x})$ is not uniquely determined. When a point approaches the interface from different sides, different limiting values are obtained for the function $a(t, \mathbf{x})$.

The kernel Φ of the operator of evaluation of the macroscopic characteristics of the composite material formulated above leads to the equality

$$\nabla \Phi(\hat{\mathbf{x}} - \mathbf{x}) = -\hat{\nabla} \Phi(\hat{\mathbf{x}} - \mathbf{x}). \quad (1)$$

Here the operators ∇ and $\hat{\nabla}$ are defined by the expressions

$$\nabla = \sum_{i=1}^3 \mathbf{i}_i \frac{\partial}{\partial x^i}, \quad \hat{\nabla} = \sum_{i=1}^3 \mathbf{i}_i \frac{\partial}{\partial \hat{x}^i},$$

where \mathbf{i}_i are the basis vectors of a Cartesian rectangular coordinate system and x^i and \hat{x}^i are the coordinates of the vectors \mathbf{x} and $\hat{\mathbf{x}}$ in the Cartesian rectangular coordinate system.

It can be shown that the fact that the kernel Φ is continuous and equal to zero at distance from the point $\hat{\mathbf{x}}$ implies validity of the condition

$$\int_{V-S} b \Phi \nabla \cdot \mathbf{c} dV = - \int_{V-S} \mathbf{c} \cdot \nabla (b \Phi) dV + \int_S \Phi \mathbf{n} \cdot (b \mathbf{c} - \bar{b} \bar{\mathbf{c}}) dS, \quad (2)$$

where $b = b(t, \mathbf{x})$ and $\mathbf{c} = \mathbf{c}(t, \mathbf{x})$ are piecewise-continuous, piecewise-smooth scalar and vector functions, respectively and $\Phi = \Phi(\hat{\mathbf{x}} - \mathbf{x})$. Integration over the surface S implies evaluation of the integral over all inner surfaces of the article. The bar indicates that the values are taken on the interface on the side of the phase for which the normal \mathbf{n} to the surface S is inward. The lack of a bar above parameters indicates that their values on the interface are taken on the side of the phase for which the normal \mathbf{n} is outward.

The physical sense of the operator of evaluation of the macroscopic characteristics reduces to spatial averaging of the structural parameter with a specified weight coefficient. The value of this coefficient depends on the distance between the point of evaluation of the integrand and the point for which the value of the macroscopic parameter is determined. The continuity and differentiability of the function Φ is required in order that the macroscopic characteristics of the composite material be continuous and differentiable as a result of action operation of the operator.

Macroscopic Law of Energy Conservation. We apply the operator of evaluation of the macroscopic characteristics of a composite material to the energy conservation law formulated at the structural level in the actual configuration:

$$\int_{V-S} \Phi \left(\frac{\partial}{\partial t} \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) + \nabla \cdot \left(\rho e \mathbf{v} + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} \right) - \nabla \cdot (\mathbf{T} \cdot \mathbf{v}) \right) dV = 0. \quad (3)$$

Here ρ is the material density, e is the mass density of the internal energy, \mathbf{v} is the velocity, and T is the Cauchy stress tensor. In what follows, the state variables ρ , e , \mathbf{v} , and T are treated as functions of the arguments t and \mathbf{x} (the actual configuration is used). If for the kernel Φ , its dependence on the argument is not indicated, it is implied that the kernel of the operator is represented as the function $\Phi = \Phi(\hat{\mathbf{x}} - \mathbf{x})$.

Expression (3) contains inner macroscopic points of the article, i.e., the points $\hat{\mathbf{x}}$ separated from the outer boundary of the article by a distance larger than a_Φ . Therefore, in evaluation of the macroscopic state variables of the medium, the effect of the boundaries of the article is ignored. The boundaries are located far enough from the examined macroscopic point at which the kernel of the integral operator Φ vanishes.

Taking into account that the kernel of the operator Φ does not depend on time and using the property (2), we obtain

$$\begin{aligned} & \int_{V-S} \frac{\partial}{\partial t} \left(\Phi \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) \right) dV - \int_{V-S} \nabla \Phi \cdot \left(\rho e \mathbf{v} + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} - T \cdot \mathbf{v} \right) dV \\ & + \int_S \Phi \mathbf{n} \cdot \left((\rho e \mathbf{v} - \bar{\rho} \bar{e} \bar{\mathbf{v}}) + \frac{1}{2} (\rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} - \bar{\rho} (\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}) \bar{\mathbf{v}}) - (T \cdot \mathbf{v} - \bar{T} \cdot \bar{\mathbf{v}}) \right) dS = 0. \end{aligned} \quad (4)$$

The time derivative of the volume integral, whose domain of integration includes discontinuity surfaces of the integrand, is determined from the expression

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\int_{V-S} \Phi \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) dV \right) = \int_{V-S} \frac{\partial}{\partial t} \left(\Phi \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) \right) dV \\ & + \int_S \Phi \left((\rho e - \bar{\rho} \bar{e}) + \frac{1}{2} (\rho (\mathbf{v} \cdot \mathbf{v}) - \bar{\rho} (\bar{\mathbf{v}} \cdot \bar{\mathbf{v}})) \right) \mathbf{n} \cdot \mathbf{v}^s dS. \end{aligned} \quad (5)$$

The time variation of the volume integral of the piecewise-continuous function is equal to the sum of the volume integral of the time derivative of the integrand and the integrals over the discontinuity surfaces of the function taking into account the magnitude of the jump and the velocity of the discontinuity surfaces in space. The form of the equation takes into account that the integrand is equal to zero at sufficient distance from the point $\hat{\mathbf{x}}$ in volume V .

Let us consider composite materials in which the binder has a rigid connection with the filling agent. This implies that the velocity of points of the inner interfaces (\mathbf{v}^s) is simultaneously the velocity of points of the binder and the velocity of points of the filling agent; i.e., on the inner boundaries, the following condition is satisfied:

$$\mathbf{v}^s = \bar{\mathbf{v}} = \mathbf{v}. \quad (6)$$

On the interface, the condition of equality of the acting forces is satisfied:

$$\mathbf{n} \cdot \bar{T} = \mathbf{n} \cdot T. \quad (7)$$

From this, taking into account relations (5)–(7), we write expression (4) as

$$\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) dV \right) - \int_{V-S} \nabla \Phi \cdot \left(\rho e \mathbf{v} + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} - T \cdot \mathbf{v} \right) dV = 0. \quad (8)$$

Using the property of the kernel of the integral operator (1), we bring (8) to the condition

$$\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) dV \right) + \hat{\nabla} \cdot \int_{V-S} \Phi \left(\rho e \mathbf{v} + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} - T \cdot \mathbf{v} \right) dV = 0, \quad (9)$$

which includes averaged structural quantities and the macroscopic nabla operator.

Requirement of Satisfaction of the Galilei Principle for the Macroscopic Energy Conservation Law. When additional uniform translational motion is imparted to all points of the article, this should not violate the energy conservation law and change in the values of the macroscopic quantities. We consider the consequences of this requirement using the results of [11]. For any constant vector \mathbf{v}_0 , we require satisfaction of the equality

$$\frac{\partial}{\partial t} \left(\int_{V-S} \Phi^* \left(\rho^* e^* + \frac{1}{2} \rho^* (\mathbf{v}^* \cdot \mathbf{v}^*) \right) dV \right) + \hat{\nabla}^* \cdot \int_{V-S} \Phi^* \left(\rho^* e^* \mathbf{v}^* + \frac{1}{2} \rho^* (\mathbf{v}^* \cdot \mathbf{v}^*) \mathbf{v}^* - T^* \cdot \mathbf{v}^* \right) dV = 0, \quad (10)$$

where $\Phi^* = \Phi(\hat{\mathbf{x}}^* - \mathbf{x})$, $\rho^*(t, \mathbf{x}) = \rho(t, \mathbf{x} - t\mathbf{v}_0)$, $e^*(t, \mathbf{x}) = e(t, \mathbf{x} - t\mathbf{v}_0)$, $T^*(t, \mathbf{x}) = T(t, \mathbf{x} - t\mathbf{v}_0)$, $\mathbf{v}^*(t, \mathbf{x}) = \mathbf{v}(t, \mathbf{x} - t\mathbf{v}_0) + \mathbf{v}_0$, $\hat{\mathbf{x}}^* = \hat{\mathbf{x}} + \mathbf{v}_0 t$, and the nabla operator is defined by the expressions

$$\hat{\nabla}^* = \sum_{i=1}^3 \mathbf{i} \frac{\partial}{\partial \hat{x}_*^i}, \quad \hat{\mathbf{x}}^* = \sum_{i=1}^3 \hat{x}_*^i \mathbf{i}.$$

Since the integration is performed over the entire volume of Euclidean space, we have

$$\int_{V-S} \Phi(\hat{\mathbf{x}}^* - \mathbf{x}) a(t, \mathbf{x} - t\mathbf{v}_0) dV = \int_{V-S} \Phi(\hat{\mathbf{x}}^* - t\mathbf{v}_0 - \mathbf{x}) a(t, \mathbf{x}) dV = \int_{V-S} \Phi(\hat{\mathbf{x}} - \mathbf{x}) a(t, \mathbf{x}) dV.$$

From the physical point of view, dependence (10) is an extension of law (9) to the case of imparting translational motion to all points of the system.

For further transformation of the macroscopic energy conservation law, we use the equalities

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_{V-S} \Phi(\hat{\mathbf{x}}^* - \mathbf{x}) a(t, \mathbf{x} - t\mathbf{v}_0) dV \right) &= \frac{\partial}{\partial t} \left(\int_{V-S} \Phi(\hat{\mathbf{x}}^* - t\mathbf{v}_0 - \mathbf{x}) a(t, \mathbf{x}) dV \right) \\ &= \frac{\partial}{\partial t} \left(\int_{V-S} \Phi(\hat{\mathbf{x}} - \mathbf{x}) a(t, \mathbf{x}) dV \right) - \mathbf{v}_0 \cdot \hat{\nabla} \int_{V-S} \Phi(\hat{\mathbf{x}} - \mathbf{x}) a(t, \mathbf{x}) dV, \\ \hat{\nabla}^* \int_{V-S} \Phi(\hat{\mathbf{x}}^* - \mathbf{x}) a(t, \mathbf{x} - t\mathbf{v}_0) dV &= \hat{\nabla} \int_{V-S} \Phi(\hat{\mathbf{x}} - \mathbf{x}) a(t, \mathbf{x}) dV, \end{aligned}$$

and write law (10), grouping terms with the same powers of the vector \mathbf{v}_0 :

$$\begin{aligned} &\frac{1}{2} \left[\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \rho dV \right) + \hat{\nabla} \cdot \int_{V-S} \Phi \rho \mathbf{v} dV \right] (\mathbf{v}_0 \cdot \mathbf{v}_0) \\ &+ \left[\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \rho \mathbf{v} dV \right) + \hat{\nabla} \cdot \int_{V-S} \Phi \rho \mathbf{v} \mathbf{v} dV - \hat{\nabla} \cdot \int_{V-S} \Phi T dV \right] \cdot \mathbf{v}_0 \\ &+ \left[\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) dV \right) - \hat{\nabla} \cdot \int_{V-S} \Phi (T \cdot \mathbf{v}) dV \right. \\ &\quad \left. + \hat{\nabla} \cdot \int_{V-S} \Phi \left(\rho e \mathbf{v} + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} \right) dV \right] = 0. \end{aligned} \quad (11)$$

This equality holds for any value of the vector \mathbf{v}_0 only if in (11) the expressions in square brackets vanish:

$$\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \rho dV \right) + \hat{\nabla} \cdot \int_{V-S} \Phi \rho \mathbf{v} dV = 0; \quad (12)$$

$$\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \rho \mathbf{v} dV \right) + \hat{\nabla} \cdot \int_{V-S} \Phi \rho \mathbf{v} \mathbf{v} dV - \hat{\nabla} \cdot \int_{V-S} \Phi T dV = 0; \quad (13)$$

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\int_{V-S} \Phi \left(\rho e + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \right) dV \right) - \hat{\nabla} \cdot \int_{V-S} \Phi (T \cdot \mathbf{v}) dV \\ &+ \hat{\nabla} \cdot \int_{V-S} \Phi \left(\rho e \mathbf{v} + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} \right) dV = 0. \end{aligned} \quad (14)$$

Determining the Macroscopic Quantities of the Material. We consider the dependences obtained. According to the physical sense of the terms, it is necessary to introduce macroscopic notions so that equalities (12)–(14) have the classic form of the law of conservation of mass

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot (\hat{\rho} \hat{\mathbf{v}}) = 0, \quad (15)$$

the equation of motion

$$\frac{\partial}{\partial t} (\hat{\rho} \hat{\mathbf{v}}) + \hat{\nabla} \cdot (\hat{\rho} \hat{\mathbf{v}} \hat{\mathbf{v}}) - \hat{\nabla} \cdot \hat{T} = 0, \quad (16)$$

and the energy conservation law

$$\frac{\partial}{\partial t} \left(\hat{\rho} \hat{e} + \frac{1}{2} \hat{\rho} (\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}) \right) + \hat{\nabla} \cdot \left(\hat{\rho} \hat{\mathbf{e}} \hat{\mathbf{v}} + \frac{1}{2} \hat{\rho} (\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} \right) - \hat{\nabla} \cdot (\hat{T} \cdot \hat{\mathbf{v}}) + \hat{\nabla} \cdot \hat{\mathbf{h}} = 0 \quad (17)$$

at the macroscopic level. The last formula contains the macroscopic energy flux $\hat{\mathbf{h}}$. Laws (15)–(17) at the macroscopic level are obtained from Eqs. (12)–(14) by introducing the following macroscopic quantities.

The macroscopic density of the material $\hat{\rho}$ at an arbitrary point of Euclidean space at a current time is reasonably determined by averaging the microstructural density ρ :

$$\hat{\rho} = \int_{V-S} \rho \Phi dV. \quad (18)$$

The macroscopic velocity of the medium should be evaluated from the condition of equality of the macroscopic and averaged structural momenta:

$$\hat{\rho} \hat{\mathbf{v}} = \int_{V-S} \rho \mathbf{v} \Phi dV. \quad (19)$$

If the macroscopic velocity is introduced by averaging the structural velocity of points of the material, the condition of objectivity of the macroscopic model will fail. The law of conservation of macroscopic mass will be violated.

According to the above definitions, the macroscopic radius vector should be evaluated from the formula

$$\hat{\mathbf{x}}(t, \hat{\mathbf{x}}_0) = \hat{\mathbf{x}}_0 + \int_{t_*}^t \hat{\mathbf{v}}(\tau, \hat{\mathbf{x}}(\tau)) d\tau,$$

where $\hat{\mathbf{x}}_0$ is the macroscopic radius vector at the current time t_0 . In contrast to the macroscopic quantities introduced previously, the radius vector is found as the radius vector of the position of points of the macroscopic continuum that corresponds to the field of macroscopic velocities rather than by averaging the corresponding structural vector over any region of space.

The macroscopic stress tensor \hat{T} of the material is introduced by averaging the structural stress tensor and the additional averaged term:

$$\hat{T} = - \int_{V-S} \Phi \rho \Delta \mathbf{v} \Delta \mathbf{v} dV + \int_{V-S} \Phi T dV.$$

Here

$$\Delta \mathbf{v} = \mathbf{v}(t, \mathbf{x}) - \hat{\mathbf{v}}(t, \hat{\mathbf{x}}). \quad (20)$$

To evaluate the macroscopic mass density of the internal energy of the material \hat{e} , it is expedient to use the following statement. The sum of the macroscopic cubic densities of the internal and kinetic energies is the sum of the corresponding averaged structural quantities:

$$\hat{\rho} \hat{e} + \frac{1}{2} \hat{\rho} \hat{\mathbf{v}} \cdot \hat{\mathbf{v}} = \int_{V-S} \left(\rho e + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \right) \Phi dV.$$

Using formulas (18)–(20), we obtain

$$\hat{\rho} \hat{e} = \int_{V-S} \left(\rho e + \frac{1}{2} \rho \Delta \mathbf{v} \cdot \Delta \mathbf{v} \right) \Phi dV.$$

In Eq. (17), the energy flux $\hat{\mathbf{h}}$ appears because the sum of the macroscopic energy fluxes and the flux of the work performed $\hat{T} \cdot \hat{\mathbf{v}}$ are equal to the corresponding averaged fluxes at the structural level:

$$\hat{\rho} \hat{\mathbf{e}} \hat{\mathbf{v}} + \frac{1}{2} \hat{\rho} (\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} - \hat{T} \cdot \hat{\mathbf{v}} + \hat{\mathbf{h}} = \int_{V-S} \Phi \left(\rho e \mathbf{v} + \frac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} - T \cdot \mathbf{v} \right) dV.$$

It is easy to see that the value of the macroscopic energy flux is defined by the expression

$$\hat{\mathbf{h}} = \int_{V-S} \Phi \left(\rho e \Delta \mathbf{v} + \frac{1}{2} \rho (\Delta \mathbf{v} \cdot \Delta \mathbf{v}) \Delta \mathbf{v} - T \cdot \Delta \mathbf{v} \right) dV.$$

In many practical problems, the value of the vector $\hat{\mathbf{h}}$ is small and is frequently set equal to zero.

The introduction of macroscopic state variables makes sense only if they can be used to construct a closed system of constitutive equations, which is not always possible. The distribution of the state variables of the material at the structural level and the averaging region (i.e., the quantity \hat{a}_Φ) should be chosen such that they allow one to establish function relations between the macroscopic parameters of the medium and to describe the macroscopic properties of the material using the corresponding equations.

Using an Averaging Operator to Analyze the Macroscopic Parameters of a Medium. We give an example of using a smoothing averaging operator. For simplicity, we restrict ourselves to analysis of a plane problem. Let us consider the problem of evaluating the macroscopic density and macroscopic stresses in a system consisting of an incompressible matrix and 91 absolutely rigid inclusions of identical radius with centers at the nodes of a regular triangular lattice (Fig. 1). At sufficient distance from the inclusions, the system is loaded by a tensile stress in the x_1 direction. The volume fraction of the filling agent in the ensemble is 0.5. The radii of the inclusions are $r_{\text{pat}} = 1$. We solve the problem for small strains. The behavior of the matrix material is assumed to obey the isotropic Hooke's law. The boundary-value problem of determining stresses at any point is solved by an iterative method using the theory of functions of a complex variable [12].

As the kernel of the averaging operator Φ , we use the function

$$\Phi(r) = \begin{cases} \gamma/a_\Phi^2, & 0 \leq r < a_\gamma, \\ \gamma[1 + \cos(\pi(r - a_\gamma)/(a_\Phi - a_\gamma))]/(2a_\Phi^2), & a_\gamma \leq r < a_\Phi, \\ 0, & a_\Phi \leq r, \end{cases}$$

where $r = \sqrt{(\hat{\mathbf{x}} - \mathbf{x}) \cdot (\hat{\mathbf{x}} - \mathbf{x})}$. The value of the parameter γ is determined from the normalization condition. An increase in the value of the argument leads to a smooth variation in the function $\Phi(r)$ from the maximal value to zero. Therefore, the points of Euclidean space located near the boundary of the domain of action of the integral operator (for which $r \approx a_\Phi$) practically do not participate in the formation of the value of the macroscopic quantity. In the calculations (unless otherwise specified) we use the value of the constant $a_\gamma = 0$.

The result of action of the averaging operator is the most obvious in evaluating the macroscopic density of the medium. We explore the behavior of the function $z = z(x_1, x_2)$, which shows how the required macroscopic density differs from the matrix density ρ_{bin} :

$$z = \Delta\rho/\Delta\rho_c.$$

Here $\Delta\rho = \hat{\rho} - \rho_{\text{bin}}$ and $\Delta\rho_c = \rho_{\text{inc}} - \rho_{\text{bin}}$; the computation result is normalized using the difference $\Delta\rho_c$ between the densities of the filling agent (ρ_{inc}) and the matrix (ρ_{bin}). Since we consider an ensemble of inclusions in which the volumetric fraction of the filling agent is 0.5, the value of the function z in the ensemble should also be equal to 0.5. This number is the control one in checking the solution obtained.

To perform calculations, it is necessary to choose a value of the parameter a_Φ . For this, we determine the values of a_Φ for which the macroscopic density of the material in the ensemble considered can be calculated with high accuracy. Figure 2a and b shows the function $z = z(x_1, x_2)$ for $a_\Phi = 5$ and 10, and Fig. 2c and d shows the constant level lines of this function with a step of 0.05 in the range of the function from 0.05 to 0.5. For $a_\Phi = 5$ (Fig. 2a), one can see a region with an effective density of the mass of the ensemble on the surface z . In this region, the value of the function is nearly equal to 0.5, which is supported by the nature of the distribution of the constant level lines (Fig. 2c). Between the region with an effective macroscopic density of the ensemble of particles and the matrix there is a transitional region which ensures a smooth transition of the properties.

The plots of the function for $a_\Phi = 10$ (Fig. 2b and d) are similar to the plots obtained for $a_\Phi = 5$. The difference is that the region with effective properties of the ensemble decreases, whereas the region with transitional

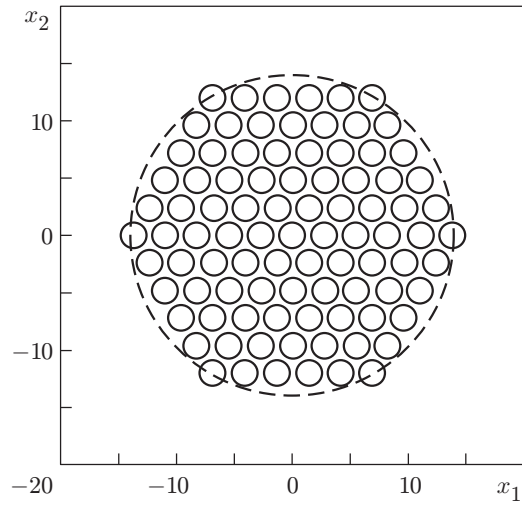


Fig. 1. Geometry of a plane composite system (the dashed curve indicates the presumed macroscopic region of space occupied by the ensemble of inclusions.)

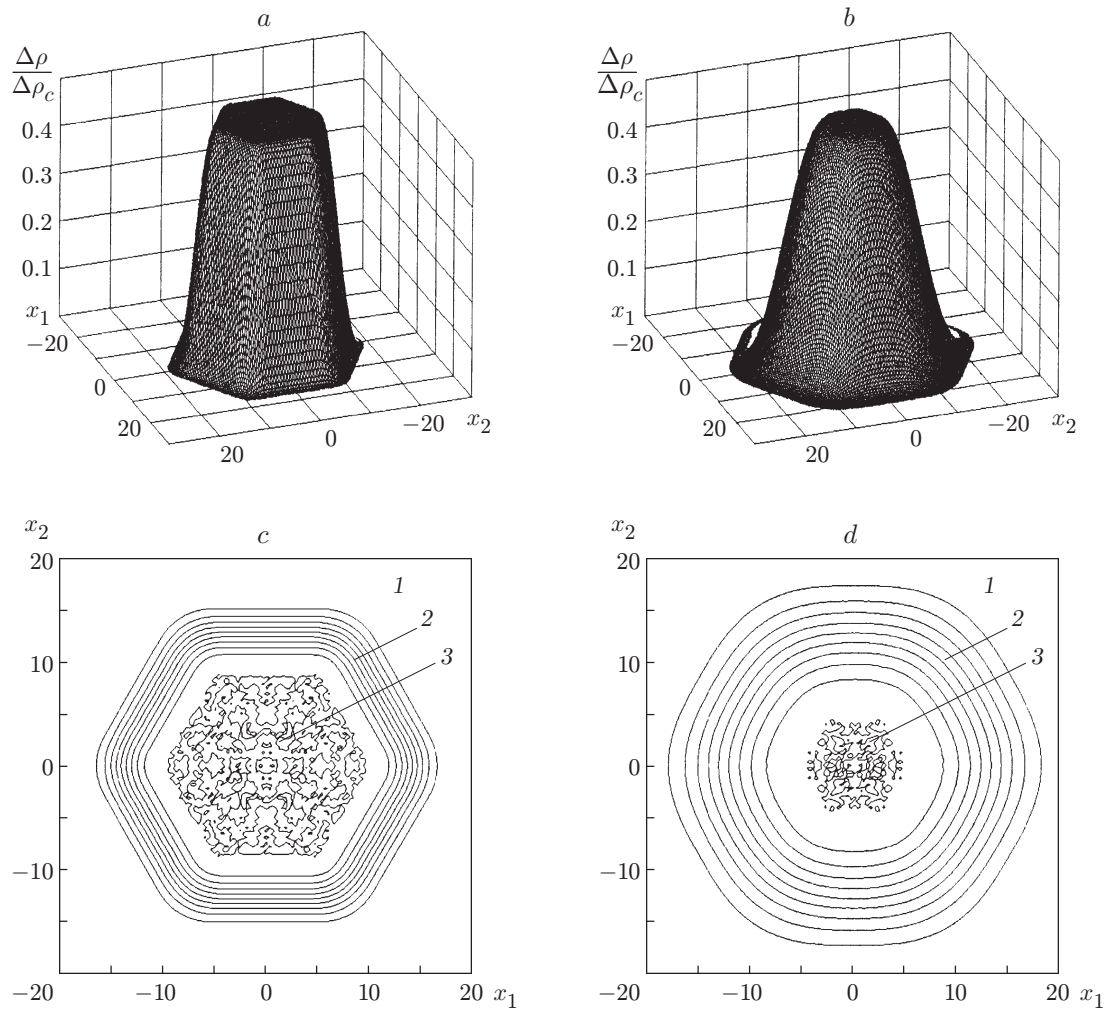


Fig. 2. Level surfaces (a and b) and lines (c and d) of the function $z = z(x_1, x_2)$ for $a_\Phi = 5$ (a and c) and 10 (b and d); 1) matrix; 2) transitional region; 3) composite.

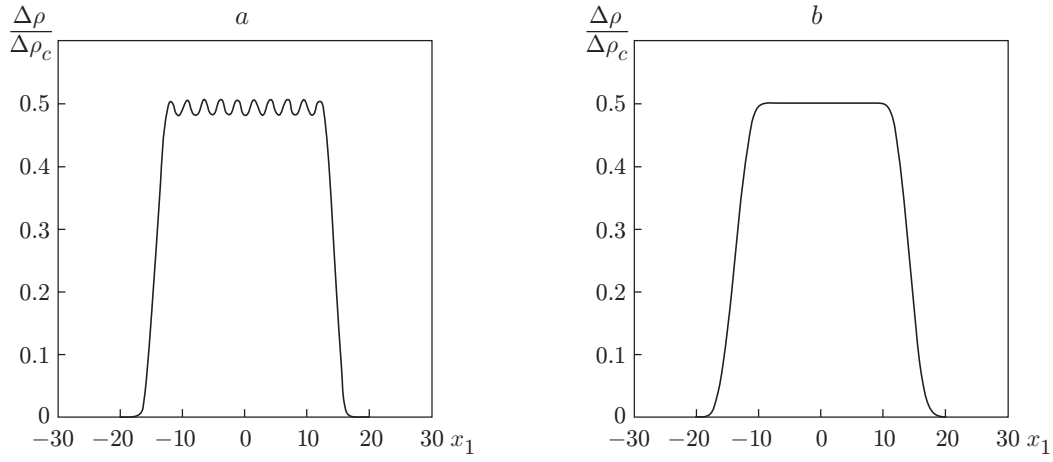


Fig. 3. Behavior of the function $z = z(x_1, 0)$ for $a_\Phi = 3$ (a) and 5 (b).

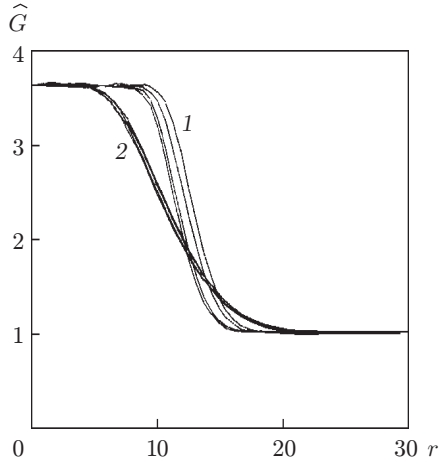


Fig. 4

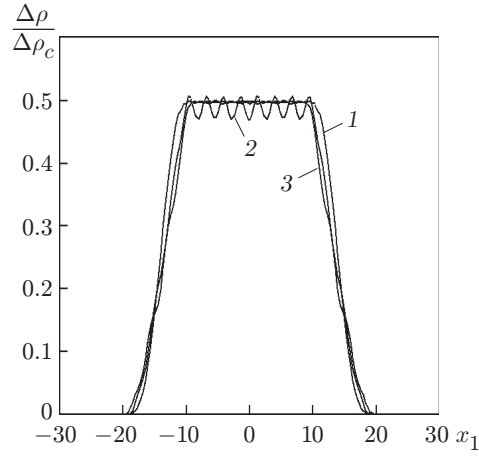


Fig. 5

Fig. 4. Distribution of the effective shear modulus in the system with an ensemble of inclusions along the r axis drawn to the x_1 axis at angles of 0, 10, 20, and 30° and $a_\Phi = 5$ (a) and 10 (b).

Fig. 5. Behavior of the function $z = z(x_1, 0)$ for $a_\gamma/a_\Phi = 0$ (1), $\sqrt{2}/2$ (2), and 0.99 (3).

properties increases considerably (Fig. 2d). In this case, the constant level lines are more similar in shape to circles rather than to regular hexagons.

The use of the integral averaging operator to derive the macroscopic properties of an ensemble of inclusions leads to the inference on the existence of a transitional region which ensures a smooth transition from the characteristics of the matrix to the effective characteristics of the ensemble.

Figure 3 shows the central sections of the surface $z = z(x_1, x_2)$ cut by the plane $x_2 = \text{const}$ for $a_\Phi = 3$ and 5. From Fig. 3 it follows that the macroscopic density in the ensemble of inclusions is determined with high accuracy for $a_\Phi > 5$. Its small deviations (less than 0.01) from the value of 0.5 are hardly seen on the plot, whereas for $a_\Phi = 3$, considerable deviations from the value of 0.5 are observed.

In the problem considered, the binder and particles of the filling agent are incompressible materials. Naturally, at the macroscopic level, the volume of the composite cannot vary. It is of interest to evaluate the macroscopic shear modulus \hat{G} . We determine it from the formula

$$2\hat{G} = \frac{\sqrt{(\hat{\sigma}_1 - \hat{\sigma}_2)^2 + (\hat{\sigma}_2 - \hat{\sigma}_3)^2 + (\hat{\sigma}_3 - \hat{\sigma}_1)^2}}{\sqrt{(\hat{\epsilon}_1 - \hat{\epsilon}_2)^2 + (\hat{\epsilon}_2 - \hat{\epsilon}_3)^2 + (\hat{\epsilon}_3 - \hat{\epsilon}_1)^2}},$$

where $\hat{\sigma}_i$ and $\hat{\epsilon}_i$ are the principal components of the macroscopic stresses and macroscopic strains in the system.

Figure 4 shows the variation of the macroscopic shear modulus \hat{G} along the r axis drawn from the coordinate origin at angles of 0, 10, 20, and 30° to the x_1 axis. It is evident that at the center of the examined ensemble of inclusions, the macroscopic shear modulus is a constant. This constant coincides with the effective modulus calculated on the periodicity cell of the composite material with the centers of inclusions at the nodes of a regular triangular lattice. At a distance from the center there is a smooth transition to the value of the shear modulus of the elastic matrix. When the value $a_\Phi = 10$ is used in the averaging operator, the dependences $\hat{G}(r)$ practically coincide (set of curves 2 in Fig. 4). This is explained by the fact that in the case considered, the shape of the region occupied by the composite material is nearly circular. When the value $a_\Phi = 5$ is used in the averaging operator, the curves are shifted relative to one another. This is due to the fact that the shape of the composite region of the system is nearly hexagonal.

Figure 5 shows plots of the function $z = z(x_1, 0)$ for various values of the parameter a_γ and a fixed value of a_Φ . For two of the examined values of a_γ/a_Φ , smooth plots of the function $z = z(x_1, x_2)$ are obtained. For $a_\Phi/a_\gamma = \sqrt{2}$, a plot is obtained whose behavior exhibits the periodicity of the structure. This example indicates that the choice of a smoothing operator and the dimension of the averaging region plays an important role and depends on the type of composite material. The criterion for the applicability of the operator is the derivation of macroscopic fields of state variables between which there are macroscopic relationships (it is possible to formulate macroscopic equations of state of the medium).

Conclusions. Using integral operators to determine the relationship between the structural and macroscopic levels of description of a composite material and the requirement of objectivity of the macroscopic formulation of the energy conservation law, one can correctly introduce the macroscopic state variables of the material taking into account finite strains of the material.

In a description at the macroscopic level, a distinct boundary between an ensemble of a finite number of inclusions and the infinite matrix surrounding this ensemble does not exist. There is a smooth transition from the effective properties of the ensemble to the properties of the matrix.

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REFERENCES

1. R. I. Nigmatulin, *Dynamics of Multiphase Media*, Part 1, Hemisphere, New York (1991).
2. L. S. Bennethum and J. H. Cushman, "Multiscale, hybrid mixture theory for swelling systems. 1. Balance laws," *Int. J. Eng. Sci.*, **34**, 125–145 (1996).
3. L. S. Bennethum, M. A. Murad, and J. H. Cushman, "Macroscale thermodynamics and the chemical potential for swelling porous media," *Transp. Porous Media*, **39**, 187–225 (2000).
4. W. G. Gray and S. M. Hassanizadeh, "Macroscale continuum mechanics for multiphase porous-media flow including phases, interfaces, common lines and common points," *Adv. Water Res.*, **21**, 261–281 (1998).
5. C. K. Chui, *An Introduction to Wavelets*, Academic Press (2001).
6. I. M. Dremin, O. V. Ivanov, and V. A. Nechitailo, "Wavelets and their use," *Usp. Fiz. Nauk.*, **171**, No. 5, 465–501 (2001).
7. J.-S. Chen, C. Pan, C.-T. Wu, and W. K. Liu, "Reproducing kernel particle methods for large deformation analysis of nonlinear structures," *Comput. Meth. Appl. Mech. Eng.*, **139**, 195–227 (1996).
8. J.-S. Chen, C. Pan, and C.-T. Wu, "Large deformation analysis of rubber based on a reproducing kernel particle method," *Comput. Mech.*, **19**, 211–227 (1997).
9. E. Hardee, K.-H. Chang, I. Grindeanu, et al. "A structural nonlinear analysis workspace (SNAW) based on meshless methods," *Adv. Eng. Software*, **30**, 153–175 (1999).
10. W. K. Liu, S. Hao, T. Belytschko, et al., "Multiple scale meshfree methods for damage fracture and localization," *Comput. Mater. Sci.*, **16**, 197–205 (1999).
11. A. E. Green, "On Cauchy's equations of motion," *Z. Angew. Math. Phys.*, **15**, 290–292 (1964).
12. A. L. Svistkov, O. K. Garshin, S. E. Evlampieva, and S. N. Lebedev, "Iterative computational method for calculating stress-strained states in ensembles of inclusions," in: *Mekh. Komposit. Mat. Strukt.*, **5**, No. 2, 17–28 (1999).